

Geometry of Kodaira moduli spaces *

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Abstract

A general theorem on the existence of natural torsion-free affine connections on a complete family of compact complex submanifolds in a complex manifold is proved.

0. Introduction. Let X be a compact complex submanifold of a complex manifold Y . The chief object of interest in this paper is the set M of all holomorphic deformations of X inside Y , i.e. a point t in M can be thought of as a "nearby" compact complex submanifold X_t in Y . The first natural question about this set — when is it a manifold? — was answered by Kodaira [9] in 1962. He found a surprisingly simple sufficient condition, the vanishing of the first cohomology group $H^1(X, N)$ with coefficients in the normal sheaf $N \equiv TY|_X / TX$, for M to have an induced complex manifold structure. Moreover, he established a clear correspondence between the *first* floors of two towers of infinitesimal neighbourhoods generated by embeddings, $t \hookrightarrow M$ and $X_t \hookrightarrow Y$, by proving that there is a *canonical* isomorphism $k_t : T_t M \longrightarrow H^0(X_t, N_t)$ which associates a global section of the normal bundle N_t of $X_t \hookrightarrow Y$ to any tangent vector at the corresponding point $t \in M$. The complete parameter space M is called a Kodaira moduli space.

About 14 years later Penrose [17] studied the Kodaira moduli space M of rational curves $X_t \simeq \mathbb{CP}^1$ embedded into a 3-dimensional complex manifold Y with normal bundle $N_t \cong \mathbb{C}^k \otimes \mathcal{O}(\mathbb{K})$. His striking conclusion was that 4-fold M comes equipped canonically with an induced half-flat conformal structure. This Penrose idea to study geometric structures canonically induced on Kodaira moduli spaces led to a radically new way of solving of a number non-linear differential equations in the form of constraints on curvature and torsion tensors of an affine connection revealing thereby deep interconnections between multidimensional complex analysis and differential geometry (see, e.g., the books [1, 3, 4, 7, 11, 18, 21] and references cited therein). Instead of analysing some particular Kodaira moduli spaces (as is normally done in twistor theory, where moduli spaces of rational curves and quadrics with specific normal bundles have been only considered), we attempt in this paper to find sufficient conditions for a general Kodaira moduli space to have a canonically induced family of torsion-free affine connections satisfying some natural integrability conditions. We do it through the analysis of the *second* floors of towers of infinitesimal neighbourhoods generated by embeddings $t \hookrightarrow M$ and $X_t \hookrightarrow Y$. Probably the most unexpected aspect of the emerging theory is that both the Kodaira condition for existence of an induced complex manifold structure and the conditions for existence

*1991 *Mathematics Subject Classification*: 32G10, 32L25, 53B05, 53B10.

of an induced geometric structure on M have the form of vanishing of some cohomology groups associated with normal bundles. As an application we show how several "curved twistor" constructions follow from one and the same general theorem thus merging several fragments into one picture.

1. Kodaira's existence theorem. Let Y and M be complex manifolds and let $\pi_1 : Y \times M \rightarrow Y$ and $\pi_2 : Y \times M \rightarrow M$ be the natural projections. An analytic family of compact submanifolds of the complex manifold Y with the moduli space M is a complex submanifold $F \hookrightarrow Y \times M$ such that the restriction of the projection π_2 on F is a proper regular map (regularity means that the rank of the differential of $\nu \equiv \pi_2|_F : F \rightarrow M$ is equal at every point to $\dim M$). Thus the family F has the structure of a double fibration

$$Y \xleftarrow{\mu} F \xrightarrow{\nu} M$$

where $\mu \equiv \pi_1|_F$. For each $t \in M$ the compact complex submanifold $X_t = \mu \circ \nu^{-1}(t) \hookrightarrow Y$ is said to belong to the family M . If $F \hookrightarrow Y \times M$ is an analytic family of compact submanifolds, then, for any $t \in M$, there is a natural linear map

$$k_t : T_t M \rightarrow H^0(X_t, N_t),$$

from the tangent space at t to the vector space of global holomorphic sections of the normal bundle $N_t = TY|_{X_t}/TX_t$ to the submanifold $X_t \hookrightarrow Y$.

An analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact submanifolds is called *complete* if the Kodaira map k_t is an isomorphism at each point t in the moduli space M . It is called *maximal at a point* $t_0 \in M$, if, for any other analytic family $\{X_{\tilde{t}} \hookrightarrow Y \mid \tilde{t} \in \tilde{M}\}$ of compact complex submanifolds such that $X_{t_0} = X_{\tilde{t}_0}$ for a point $\tilde{t}_0 \in \tilde{M}$, there exists a neighbourhood $\tilde{U} \subset \tilde{M}$ of \tilde{t}_0 and a holomorphic map $f : \tilde{U} \rightarrow M$ such that $f(\tilde{t}_0) = t_0$ and $X_{f(\tilde{t})} = X_{\tilde{t}}$ for each $\tilde{t} \in \tilde{U}$. The family $\{X_t \hookrightarrow Y \mid t \in M\}$ is called *maximal* if it is maximal at each point t in the moduli space M .

In 1962 Kodaira proved that *if $X \hookrightarrow Y$ is a compact complex submanifold with normal bundle N such that $H^1(X, N) = 0$, then X belongs to the complete analytic family $\{X_t : t \in M\}$ of compact submanifolds X_t of Y . The family is maximal and its moduli space is of complex dimension $\dim_{\mathbb{C}} H^0(X, N)$.*

2. Alpha subspaces of moduli spaces. Let $\{X_t \hookrightarrow Y \mid t \in M\}$ be a complete family of compact complex submanifolds. For any point $y \in Y' \equiv \cup_{t \in M} X_t$, there is an associated subset $\nu \circ \mu^{-1}(y)$ in M . It is easy to show that such a subset is always an analytic subspace of M . Moreover, if the natural evaluation map

$$H^0(X_t, N_t) \otimes \mathcal{O}_{X_t} \rightarrow N_t$$

is an epimorphism for all $t \in \nu \circ \mu^{-1}(y)$, then the subspace $\nu \circ \mu^{-1}(y) \subset M$ has no singularities, i.e. it is a submanifold. In general, we denote the manifold content of $\nu \circ \mu^{-1}(y)$ by α_y and, following the terminology of the seminal paper by Penrose [17], call it an *alpha subspace* of M .

3. Two natural sheaves on moduli space. Let $F \hookrightarrow Y \times M$ be a complete analytic family of compact submanifolds. The Kodaira theorem asserts the existence of a sheaf isomorphism $k : TM \longrightarrow \nu_*^0(N_F)$, where N_F denotes the normal bundle of F in $Y \times M$. It is easy to see that k^{-1} induces canonically the following morphisms of \mathcal{O}_M -modules

$$\begin{aligned} \phi_1 : \nu_*^0(N_F \otimes S^2(N_F^*)) &\longrightarrow TM \otimes S^2(\Omega^1 M), \\ \phi_2 : \nu_*^0(N_F \otimes N_F^*) &\longrightarrow TM \otimes \Omega^1 M, \end{aligned}$$

whose images we denote by Λ and Θ respectively. Thus, the locally free sheaves $TM \otimes S^2\Omega^1 M$ and $TM \otimes \Omega^1 M$ on any Kodaira moduli space M come equipped canonically with \mathcal{O}_M -submodules Λ and, respectively, Θ which may fail to be locally free in general.

4. Vector bundles on Y and affine connections on M . Let $\{X_t \hookrightarrow Y \mid t \in M\}$ be a complete family of compact complex submanifolds. After the work by Ward [19] on instanton solutions of Yang-Mills equations much attention has been paid to holomorphic vector bundles E on Y which are trivial on submanifolds X_t for all $t \in M$. Their geometric role is that they generate vector bundles on M together with linear connections which are integrable on alpha subspaces of M (see, e.g., [11, 21]). In this subsection we find a geometric meaning of holomorphic vector bundles on Y which, when restricted to X_t , $t \in M$, are canonically isomorphic to the normal bundle N_t of the embedding $X_t \hookrightarrow Y$, i.e. $E|_{X_t} = N_t$.

Theorem 1 *Let $\{X_t \hookrightarrow Y \mid t \in M\}$ be a complete family of compact submanifolds. Suppose there is a holomorphic vector bundle E on Y such that $E|_{X_t} = N_t$ for all t in some domain $M_0 \subseteq M$. If $H^0(X_t, N_t \otimes S^2(N_t^*)) = H^1(X_t, N_t \otimes S^2(N_t^*)) = 0$ for each $t \in M_0$, then M_0 comes equipped canonically with an induced torsion-free affine connection such that, for every $y \in Y'$, the associated alpha subspace $\alpha_y \cap M_0$ is totally geodesic.*

Let (X, \mathcal{O}_X) be an analytic subspace of a complex manifold (Y, \mathcal{O}_Y) defined by the sheaf of ideals $J \subset \mathcal{O}_Y$. The m th-order infinitesimal neighborhood of X in Y is the ringed space $X^{(m)} = (X, \mathcal{O}_X^{(m)})$ with the structure sheaf $\mathcal{O}_X^{(m)} = \mathcal{O}_Y/J^{m+1}$. With the $(m+1)$ th-order infinitesimal neighborhood there is naturally associated an m th-order conormal sheaf of $\mathcal{O}_X^{(m)}$ -modules $\mathcal{O}_X^{(m)}(N^*) = J/J^{m+2}$ (cf. [6]). By construction, $\mathcal{O}_X^{(0)}(N^*)$ is the usual conormal sheaf N^* , while $\mathcal{O}_X^{(1)}(N^*)$ fits into the exact sequence of $\mathcal{O}_X^{(1)}$ -modules

$$0 \longrightarrow S^2(N^*) \longrightarrow \mathcal{O}_X^{(1)}(N) \longrightarrow N^* \longrightarrow 0.$$

When X is a point in Y , then $(\mathcal{O}_X^{(0)}(N^*))^*$ is identical to the tangent space $T_X Y$ at $X \in Y$, while $(\mathcal{O}_X^{(1)}(N^*))^*$ is usually denoted by $T_X^{[2]} Y$ and called second-order tangent space.

Proof of Theorem 1. By assumption, there is a holomorphic vector bundle on Y such that $E|_{X_t} = N_t$ for any given $t \in M$. The restriction of its dual to the first order infinitesimal neighbourhood of X_t in Y is an $\mathcal{O}_{X_t}^{(1)}$ -module which fits into the exact sequence [11]

$$0 \longrightarrow N_t^* \otimes E^*|_{X_t} \xrightarrow{i} E^*|_{X_t^{(1)}} \longrightarrow E^*|_{X_t} \longrightarrow 0.$$

Therefore, one obtains an extension

$$0 \longrightarrow S^2(N_t^*) \longrightarrow E^*|_{X_t^{(1)}} / i(\wedge^2 N^*) \longrightarrow N_t^* \longrightarrow 0.$$

It is straightforward to show using local coordinate representations of all objects involved [13], that the difference

$$\left[\mathcal{O}_{X_t}^{(1)}(N^*) - E^*|_{X_t^{(1)}} / i(\wedge^2 N^*) \right] \in \text{Ext}_{\mathcal{O}_X^{(1)}}(N^*, S^2(N^*))$$

is a locally free \mathcal{O}_{X_t} -module which compares to 2nd order two embeddings, $X_t \hookrightarrow Y$ and $X_t \hookrightarrow E|_{X_t}$, which are equivalent at 0th and 1st orders. We denote its dual sheaf by $\Delta_{X_t}^{[2]}(Y, E)$. The crucial property of this creature is that there is a commutative diagram,

$$\begin{array}{ccccccc} 0 \rightarrow & T_t M & \rightarrow & T_t^{[2]} M & \rightarrow & S^2(T_t M) & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & H^0(X_t, N_t) & \rightarrow & H^0(X_t, \Delta_{X_t}^{[2]}(Y, E)) & \rightarrow & H^0(X_t, S^2(N_t)) & \end{array} \quad (1)$$

which relates the first floors of two towers of infinitesimal neighbourhoods generated by embeddings $t \hookrightarrow M_0$ and $X_t \hookrightarrow Y$. The verification is straightforward when one uses local coordinates [13]. Since, by assumption, $H^1(X_t, N_t \otimes S^2(N^*)) = 0$, the exact sequence

$$0 \longrightarrow N_t \longrightarrow \Delta_{X_t}^{[2]}(Y, E) \longrightarrow S^2(N_t) \longrightarrow 0 \quad (2)$$

splits. Moreover, its splitting is unique for the group $H^0(X_t, N_t \otimes S^2(N_t^*))$ is also assumed to vanish. The commutative diagram (1) implies then a canonical splitting of the exact sequence for $T_t^{[2]} M$ which is the same thing as a torsion-free affine connection at t . This shows that the moduli space M_0 comes equipped canonically with an induced torsion-free affine connection. The fact that this connection is integrable on alpha surfaces is an easy exercise. \square

For any given relative deformation problem $X \hookrightarrow Y$, it is not difficult to identify a large class of holomorphic vector bundles E on the ambient manifold Y which have the property required by Theorem 1. Indeed, take any holomorphic distribution $E \subset TY$ on Y which is transverse to X and has rank equal to $\text{codim } X$. Then, for all t in a sufficiently small neighbourhood M_0 of the point $t_0 \in M$ corresponding to X , the submanifolds $X_t \hookrightarrow Y$ remain transverse to E , and one has a *canonical* isomorphism $E|_{X_t} = N_{X_t}$.

Let us illustrate Theorem 1 by examples. The first one is just the original Penrose non-linear graviton construction [17]. Recall that Penrose established a one-to-one correspondence between “small” 4-dimensional complex Riemannian manifolds M_0 with self-dual Riemann tensor and 3-dimensional complex manifolds Y equipped with the following data: (i) a submanifold $X \simeq \mathbb{CP}^k$ with normal bundle $N \simeq \mathbb{C}^k \otimes \mathcal{O}_{\mathbb{CP}^k}(k)$; (ii) a fibration $\pi : Y \longrightarrow X$; (iii) a global non-vanishing section of the “twisted” determinant bundle $\det(V_\pi) \otimes \pi^*(\mathcal{O}(2))$, where V_π is the vector bundle of π -vertical vector fields. Let us look at this data from the point of view of Theorem 1. The distribution V_π is clearly transversal to X , and one easily checks that $H^0(X_t, N_t \otimes S^2(N_t^*)) = H^1(X_t, N_t \otimes S^2(N_t^*)) = 0$. Therefore, by Theorem 1, the data (i) and (ii) imply that there is a torsion-free affine connection induced on M_0 which satisfies natural integrability conditions. What is the role of datum (iii)? It will be shown in section 6 that datum (iii) ensures that this induced connection is precisely the Levi-Civita connection of a holomorphic metric with self-dual Riemann tensor. Therefore, the non-linear graviton construction is one of the manifestations of the phenomenon envisaged by Theorem 1. Another manifestation is provided by relative deformation problems studied in the context of quaternionic geometry. Let Y be a complex

$(2n + 1)$ -dimensional manifold equipped with a holomorphic contact structure which is a maximally non-degenerate rank $2n$ holomorphic distribution $D \subset TY$. Let X be a rational curve embedded into Y transversely to D and with normal bundle $N = \mathbb{C}^{2n} \otimes \mathcal{O}(1)$. Then Theorem 1 says that the $4n$ -dimensional Kodaira moduli space M_0 comes equipped with a unique torsion-free affine connection satisfying natural integrability conditions. This is in accordance with well-known results in twistor theory. The case $n = 1$ has been studied by Ward [20] and Hitchin [8] who showed that M_0 has an induced complex Riemannian metric satisfying self-dual Einstein equations with non-zero scalar curvature. The case $n \geq 2$ has been investigated by Bailey and Eastwood [2], LeBrun [10], and Pedersen and Poon [15] who proved that the moduli space M_0 comes equipped canonically with a torsion-free connection compatible with the induced (complexified) quaternionic structure on M .

5. Families of affine connections on moduli space. What happens when $H^0(X_t, N_t \otimes S^2(N_t^*)) \neq 0$? In this case, as we noted in section 3, the locally free \mathcal{O}_M -module $TM \otimes S^2(\Omega^1 M)$ comes equipped with a non-zero \mathcal{O}_M -submodule Λ . If $\nabla_1 : TM \rightarrow TM \otimes \Omega^1 M$ and $\nabla_2 : TM \rightarrow TM \otimes \Omega^1 M$ are two torsion-free affine connections on M , then their difference, $\nabla_1 - \nabla_2$, is a global section of $TM \otimes S^2(\Omega^1 M)$. We say that ∇_1 and ∇_2 are Λ -equivalent, if

$$\nabla_1 - \nabla_2 \in H^0(M, \Lambda).$$

We define a Λ -connection on M as a collection of ordinary torsion-free affine connections $\{\nabla_i \mid i \in I\}$ on an open covering $\{U_i \mid i \in I\}$ of M which, on overlaps $U_{ij} = U_i \cap U_j$, have their differences in $H^0(U_{ij}, \Lambda)$. Locally, a Λ -connection is the same thing as a Λ -equivalence class of affine connections, but globally they are different — the obstruction for existence of a Λ -connection on M lies in $H^1(M, TM \otimes S^2(\Omega^1 M)/\Lambda)$, while the obstruction for existence of a Λ -equivalence class of affine connections is an element of $H^1(M, TM \otimes S^2(\Omega^1 M))$. A submanifold of M is said to be totally geodesic relative to a Λ -connection if it is totally geodesic relative to *each* of its local representatives ∇_i .

Theorem 2 *Let $\{X_t \hookrightarrow Y \mid t \in M\}$ be a complete family of compact submanifolds. Suppose there is a holomorphic vector bundle E on Y such that $E|_{X_t} = N_t$ for all t in some domain $M_0 \subseteq M$. If $H^1(X_t, N_t \otimes S^2(N_t^*)) = 0$ for each $t \in M_0$, then M_0 comes equipped canonically with an induced Λ -connection such that, for every $y \in Y'$, the associated alpha subspace $\alpha_y \cap M_0$ is totally geodesic.*

Proof. Since $H^1(X_t, N_t \otimes S^2(N_t^*))$, the exact sequence (2) splits, and any such splitting, i.e. an \mathcal{O}_{X_t} -morphism $\Delta_{X_t}^{[2]}(Y, E) \rightarrow N_t \oplus S^2(N_t^*)$, induces an associated (via commutative diagram (1)) splitting of the exact sequence for the second-order tangent space $T_t^{[2]}M$ which in turn induces a torsion-free affine connection at $t \in M_0$. The set of all splittings of (2) is a principal homogeneous space for the group $H^0(X_t, N_t \otimes S^2(N_t^*))$. Therefore, the set of induced torsion-free affine connections at t is a principal homogeneous space for the group Λ_t . \square

Next we study another equivalence class of affine connections which is often induced on Kodaira moduli spaces. Let sym denote the natural projection

$$\text{sym} : TM \otimes \Omega^1 M \otimes \Omega^1 M \rightarrow TM \otimes S^2(\Omega^1 M).$$

Associated with a subsheaf $\Xi \subset TM \otimes \Omega^1 M$ there is a concept of Ξ -connection on M which is, by definition, a collection of ordinary torsion-free affine connections $\{\nabla_i \mid i \in I\}$ on an open covering $\{U_i \mid i \in I\}$ of M which, on overlaps $U_{ij} = U_i \cap U_j$, have their differences in $H^0(U_{ij}, \text{sym}(\Xi \otimes \Omega^1 M))$. If Ξ happens to be the structure sheaf \mathcal{O}_M embedded diagonally into $TM \otimes \Omega^1 M$, then a Ξ -connection is nothing but a torsion-free projective connection on M .

Theorem 3 *Let $\{X_t \hookrightarrow Y \mid t \in M\}$ be a complete family of compact submanifolds. Suppose there is a holomorphic vector bundle E on Y such that $E|_{X_t} = N_t$ for all t in some domain $M_0 \subseteq M$. If $H^1(X_t, N_t \otimes N_t^*) = 0$ for each $t \in M_0$, then M_0 comes equipped canonically with an induced $\Lambda_{H^0(X, N \otimes N^*)}$ -connection such that, for every $y \in Y'$, the associated alpha subspace $\alpha_y \cap M_0$ is totally geodesic.*

This theorem is proved by a slight modification of the construction used in the proof of Theorem 1 [13]. Note that the condition $H^1(X_t, N_t \otimes N_t^*) = 0$ only says that N_t is a rigid vector bundle on X_t . There is an important case when the requirement $E|_{X_t} = N_t$ can be safely replaced by a much weaker condition that vector bundles $E|_{X_t}$ and N_t are simply equivalent to each other, $E|_{X_t} \simeq N_t$, which means that they define the same cohomology class in $H^1(X_t, GL(p, \mathcal{O}_{X_t}))$, $p = \text{codim} X$.

Theorem 4 *Let $\{X_t \hookrightarrow Y \mid t \in M\}$ be a complete family of compact 1-codimensional submanifolds. Suppose there is a holomorphic line bundle E on Y such that $E|_{X_t} \simeq N_t$ for all t in some domain $M_0 \subseteq M$. If $H^1(X_t, \mathcal{O}_{X_t}) = 0$ for each $t \in M_0$, then M_0 comes equipped canonically with an induced torsion-free projective connection such that, for every $y \in Y'$, the associated alpha subspace $\alpha_y \cap M_0$ is totally geodesic.*

Proof. Fix any isomorphism $\phi : E|_{X_t} \longrightarrow N_t$ and apply Theorem 4 to construct a projective connection ∇ on M_0 . Due to compactness of X_t , the isomorphism ϕ is defined up to multiplication by a non-zero constant. The point is that the construction of ∇ scetched in the proof of Theorem 1 is invariant under such a transformation which shows that ∇ is actually independent on a particular choice of the isomorphism ϕ used in its construction. \square

Corollary 5 *Let $\{X_t \hookrightarrow Y \mid t \in M\}$ be a complete family of compact 1-codimensional submanifolds. If $H^1(X_{t_0}, \mathcal{O}_{X_{t_0}}) = 0$ for some $t_0 \in M$, then there is an open neighbourhood M_0 of t_0 which comes equipped canonically with an induced torsion-free projective connection such that, for every $y \in Y'$, the associated alpha subspace $\alpha_y \cap M_0$ is totally geodesic.*

Proof. By semi-continuity principle, there is an open neighbourhood U of t_0 such that $H^1(X_t, \mathcal{O}_{X_t}) = 0$ for each $t \in U$. If L is the divisor line bundle of X_{t_0} , then $L^*|_{X_t} \simeq N_t$ for all t in probably smaller neighbourhood M_0 . This fact combined with Theorem 4 implies the desired result. \square

An independent (and more effective from the viewpoint of computing induced projective structures) proof of this Corollary is given in [14].

6. Fibred complex manifolds. Suppose that the ambient manifold Y has the structure of a holomorphic fibration over its compact submanifold X , i.e. there is a submersive holomorphic map $\pi : Y \rightarrow X$. If $H^1(X, N) = 0$, then, by Kodaira's theorem, X belongs to the complete family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact submanifolds. The submanifold X is transverse to the distribution V_π of π -vertical vector fields on Y , and so is X_t for every t in some neighbourhood M_0 of the point $t_0 \in M$ which corresponds to X . Thus Y has the structure of a holomorphic fibration $Y \rightarrow X_t$ for $t \in M_0$. If also $H^1(X_t, N_t \otimes S^2(N_t^*)) = 0$, then, by Theorem 2, the moduli space comes equipped with a family of induced torsion-free affine connections. Holonomy groups of these induced connections can be estimated with surprising ease.

Theorem 6 *Let ∇ be an induced connection on the moduli space M_0 . If the function*

$$f : t \rightarrow \dim H^0(X_t, N_t \otimes N_t^*)$$

is constant on M_0 , then the holonomy algebra of ∇ is a subalgebra of the finite dimensional Lie algebra $H^0(X, N \otimes N^)$. If in addition there is a holomorphic line bundle L on X such that the bundle $\pi^*(L) \otimes \det V_\pi^*$ admits a nowhere vanishing holomorphic section, then the holonomy algebra of ∇ is a subalgebra of $H^0(X, N \otimes_0 N^*)$, where \otimes_0 denotes trace-free tensor product.*

This theorem follows from the correspondence between *third* floors of the two towers of infinitesimal neighbourhoods of embeddings $t \hookrightarrow M$ and $X_t \hookrightarrow Y$ which has been studied in [12].

Examples. 1. Let $X = \mathbb{CP}^k$ be the projective line embedded into a 3-dimensional complex manifold Y with normal bundle $N \simeq \mathbb{C}^k \otimes \mathcal{O}(k)$. If Y has the structure of a holomorphic fibration over X , then Theorem 1 says that there is an induced connection ∇ on the moduli space M_0 . By Theorem 6, the holonomy algebra of ∇ is contained in $H^0(X, N \otimes N^*) = M_{2,2}(\mathbb{C}) \subset \times(\not\leq, \mathbb{C})$, where $M_{2,2}(\mathbb{C})$ is the space of 2×2 complex matrices and $co(4, \mathbb{C}) = \sim\triangleleft(\not\leq, \mathbb{C}) + \sim\triangleleft(\not\leq, \mathbb{C}) + \mathbb{C}$ the complexified conformal algebra. Since ∇ is torsion-free, this fact implies that ∇ is a complex Weyl connection on the 4-dimensional complex conformal manifold M_0 which has the anti-self-dual parts of the Weyl tensor and the antisymmetrized Ricci tensor vanishing and satisfies the Einstein-Weyl equations. In fact any such connection arises locally in this way [17, 5, 16].

2. Let the pair $(X = \mathbb{CP}^k, \mathbb{Y})$ be the same as in Example 1 and assume that the bundle $\det V_\pi^* \otimes \pi^*(\mathcal{O}(2))$ admits a nowhere vanishing holomorphic section. By Theorem 6, the holonomy group of the induced connection ∇ on the 4-dimensional parameter space M_0 is a subgroup of $SL(2, \mathbb{C}) \subset SO(\not\leq, \mathbb{C}) = SL(\not\leq, \mathbb{C}) \times_{\mathbb{Z}_2} SL(\not\leq, \mathbb{C})$. This means that ∇ is the Levi-Civita connection of a complex Riemannian metric on M_0 which is Ricci-flat and has the anti-self-dual part of the Weyl tensor vanishing. Again any such connection arises locally in this way [17].

3. Let $X = \mathbb{CP}^k$ be the projective line embedded into a $(2k+1)$ -dimensional complex manifold Y with normal bundle $N \simeq \mathbb{C}^{k-1} \otimes \mathcal{O}(k)$, $k \geq 2$. The Kodaira moduli space M is then a $4k$ -dimensional complex manifold possessing a complexified almost quaternionic structure. If Y has the structure of a holomorphic fibration over X , then, by Theorems 1 and 6, there is an induced connection ∇ on $M_0 \subset M$ with holonomy in $GL(2k, \mathbb{C})$ which implies that ∇ is a complexified Obata connection. It is not difficult to show using results of Bailey and Eastwood [2] that any local complexified Obata connection can be constructed in this way. A twistor characterization of real analytic Obata connections can be obtained from this holomorphic picture in the usual way (cf. [11]).

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